Extended Mean Field Games: why are they so different?

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***Jimmy* 60 th!**
October 14, 2017

Game Theory

Long History (e.g. long list of **Nobel Prizes in Economics**)

Of relevance to this presentation:

Mathematical Theory

- Notion of Equilibrium & Nash Equilibrium

Applications in Finance / Finance Engineering

- Auction Theory (IPOs, pricing of digital goods, ...)
- Predatory Trading
- Optimal Execution / Liquidation

Applications from Social Sciences & Regulation

- Large populations behavior (herding, congestion, ...)
- Games of Timing (Bank runs, Fund Redemption, ...)

Will Focus on Mean Field Games

Lasry-Lions, Caines-Huang-Malhamé, 2006

AGENT BASED MODELS AND MEAN FIELD GAMES

- **Agent Based Models**
  - Very popular for the analysis of **large** complex **systems**
  - Behavior prescribed at the individual (**microscopic**) level
  - Exogenously specified **interactions**
  - Large scale **simulations** possible
- If **symmetries** in the system, interactions can be **Mean Field**
  - Possible **averaging** effects for large populations
  - **Mean Field** limits easier to simulate and study
  - Net result: **Macroscopic** behavior of the system

Mean Field Games vs Agent Based Models

- **Mean Field Games**
  - At the (**microscopic**) level individuals control their states
  - **Exogenously** specified interaction rules
  - Individuals are rational: they **OPTIMIZE** !!!!
- Search for **equilibria**: very difficult, NP hard in general
- If **symmetries** in the system, interactions can be **Mean Field**
  - Possible **averaging** effects for large populations
  - **Mean Field** limits easier to study
  - **Macroscopic** behavior of the system thru solutions of

    Mean Field Games

    Lasry-Lions (MFG) **Caines-Huang-Malhamé** (NCE)
  - **Examples:** flocking, schooling, herding, crowd behavior, percolation of information, price formation, hacker behavior and cyber security, ...

Two Simple Examples

- **Markets for Exhaustible Resources** (e.g. crude oil)
  Guéant - Lasry - Lions and **Chan-Sircar**

- **Price Impact of a Large Group of Traders**
  R.C. - Lacker (weak formulation) **R.C. - Aghbad** (unpublished) **R.C. - Delarue** (Vol. I of big book), later extended in **Jaimungal - Nourian** & **Cardaliaguet - L Halle**

Price Impact of a Group of N Traders

X_t^i **number of shares** owned at time t, α_t^i *rate of trading* of i-th trader

$$dX_t^i = \alpha_t^i \, dt + \sigma^i dW_t^i$$

K_t^i amount of **cash** held by trader i at time t

$$dK_t^i = -[\alpha_t^i S_t + c(\alpha_t^i)] \, dt,$$

where S_t **price** of one share, $\alpha \to c(\alpha) \geq 0$ *cost for trading at rate* α

Price impact formula:

$$dS_t = \frac{1}{N} \sum_{i=1}^{N} h(\alpha_t^i) \, dt + \sigma_0 dW_t^0$$

Trader i tries to minimize

$$J^i(\boldsymbol{\alpha}^1, ..., \boldsymbol{\alpha}^N) = \mathbb{E}\Big[\int_0^T c_X(X_t^i) dt + g(X_T^i) - V_T^i \Big]$$

where V_t^i is the **wealth** of trader i at time t: $V_t^i = K_t^i + X_t^i S_t$.

Mean Field Game Formulation

Recall that $V_t^i = K_t^i + X_t^i S_t$. so that

$$
\begin{aligned}
dV_t^i &= dK_t^i + X_t^i\, dS_t + S_t\, dX_t^i \\
&= \Big[-c(\alpha_t^i) + X_t^i \frac{1}{N} \sum_{j=1}^N h(\alpha_t^j) \Big] dt + \sigma S_t dW_t^i + \sigma_0 X_t^i dW_t^0 .
\end{aligned} \tag{1}
$$

so that:

$$
J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\Big[\int_0^T f(t, X_t^i, \overline{\theta}_t^N, \alpha_t^i) dt + g(X_T^i) \Big], \tag{2}
$$

where $\overline{\theta}_t^N$ is the empirical distribution of the N controls α_t^i and:

$$
f(t, x, \theta, \alpha) = c(\alpha) + c_X(x) - x\langle h, \theta \rangle, \tag{3}
$$

for $0 \le t \le T$, $x \in \mathbb{R}^d$, $\theta \in \mathcal{P}(A)$, and $\alpha \in A$.

MFG Formulation: For each deterministic flow $\boldsymbol{\theta} = (\theta_t)_{t \ge 0}$ of probability measures on the space A of controls, solve the standard optimal control problem

$$
\begin{cases}
\inf_{\boldsymbol{\alpha}} \mathbb{E}\Big[\int_0^T f(t, X_t, \theta_t, \alpha_t) dt + g(X_T) \Big] \\
dX_t = \alpha_t dt + \sigma dW_t, \quad t \in [0, T],
\end{cases} \tag{4}
$$

for a given Wiener process $\mathbf{W}$ and find a flow of measures $\boldsymbol{\theta} = (\theta_t)_{t \ge 0}$ so that $\theta_t = \mathcal{L}(\hat{\alpha}_t)$ where $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_t)_{0 \le t \le T}$ is an optimal control for the above problem.

Disclaimer

"Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different."

Johann Wolfgang von Goethe

Assume Mean Field Interactions (symmetric game)

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dt + \sigma(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dW_t^i \quad i = 1, \cdots, N$$

Assume player i tries to minimize

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left[\int_0^T f(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dt + g(X_T, \overline{\mu}^N_{X_T^N})\right]$$

Search for Nash equilibria

- Very difficult in general, even if N is small
- ϵ-Nash equilibria? Still hard.
- How about in the limit $N \to \infty$?

Mean Field Games Lasry - Lions, Caines-Huang-Malhamé

N-Player Stochastic Differential Games

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Mean Field Games Lasry - Lions, Caines-Huang-Malhamé

MFG Paradigm

A **typical** agent plays against a **field** of players whose states he/she feels through the statistical distribution **distribution** μ_t of their states at time t

1. For each **Fixed** measure flow $\boldsymbol{\mu} = (\mu_t)$ in $\mathcal{P}(\mathbb{R})$, solve the **standard stochastic control problem**
$$\hat{\boldsymbol{\alpha}} = \arg \inf_{\boldsymbol{\alpha} \in \mathbb{A}} \mathbb{E} \left\{ \int_0^T f(t, X_t, \mu_t, \alpha_t) dt + g(X_T, \mu_T) \right\}$$
subject to
$$dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t$$
2. **Fixed Point Problem**: determine $\boldsymbol{\mu} = (\mu_t)$ so that
$$\forall t \in [0, T], \quad \mathcal{L}(X_t^{\hat{\boldsymbol{\alpha}}}) = \mu_t.$$

$\boldsymbol{\mu}$ or $\hat{\boldsymbol{\alpha}}$ is called a solution of the MFG.

Once this is done one expects that, if $\hat{\alpha}_t = \phi(t, X_t)$,
$$\alpha_t^{j*} = \phi^*(t, X_t^j), \qquad j = 1, \cdots, N$$
form an **approximate Nash equilibrium** for the game with N players.

Extended Mean Field Games

(I) For each **fixed** deterministic continuous flow $\boldsymbol{\nu} = (\nu_t)_{0 \le t \le T}$ in $\mathcal{P}(\mathbb{R}^d \times A)$, solve the **standard stochastic control problem**:

$$\inf_{\boldsymbol{\alpha} \in \mathbb{A}} J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}),$$

with

$$J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T f(t, X_t^{\boldsymbol{\alpha}}, \nu_t, \alpha_t)dt + g(X_T^{\boldsymbol{\alpha}}, \mu_T)\right],$$

subject to

$$dX_t^{\boldsymbol{\alpha}} = b(t, X_t^{\boldsymbol{\alpha}}, \nu_t, \alpha_t)dt + \sigma(t, X_t^{\boldsymbol{\alpha}}, \nu_t, \alpha_t)dW_t, \quad t \in [0, T],$$

and $X_0^{\boldsymbol{\alpha}} = \xi$ where μ_s denotes the first marginal of ν_s

(II) Find a flow $\boldsymbol{\nu} = (\nu_t)_{0 \le t \le T}$ so that, for all $t \in [0, T]$,

$$\mathcal{L}(\hat{X}_t^{\boldsymbol{\nu}}, \hat{\alpha}_t^{\boldsymbol{\nu}}) = \nu_t$$

if $\hat{\boldsymbol{\alpha}}^{\boldsymbol{\nu}} \in \mathbb{A}$ is a minimizer of $J^{\boldsymbol{\nu}}$ with $\hat{\mathbf{X}}^{\boldsymbol{\nu}}$ as optimal path.

Revisiting the Analytic Approach

Assume σ is independent of the control, and use the **reduced Hamiltonian**

$$H(t,x,\nu,y,\alpha) = b(t,x,\nu,\alpha)\cdot y + f(t,x,\nu,\alpha),$$

for $t \in [0,T]$, $x, y \in \mathbb{R}^d$, $\alpha \in A$ and $\nu \in \mathcal{P}(\mathbb{R}^d \times A)$.
Add some **convexity** assumption to find a unique minimizer

$$\hat{\alpha}(t,x,\nu,y) = \mathrm{argmin}_{\alpha \in A} H(t,x,\nu,y,\alpha)$$

HJB equation (recall, $\boldsymbol{\nu} = (\nu_t)_{0\le t\le T}$ is fixed)

$$\partial_t V(t,x) + \frac{1}{2}\mathrm{trace}\Big[\big(\sigma\sigma^\dagger\big)(t,x,\nu_t)\partial^2_{xx}V(t,x)\Big] \\ + H\Big(t,x,\nu_t,\partial_x V(t,x), \hat{\alpha}\big(t,x,\nu_t,\partial_x V(t,x)\big)\Big) = 0,$$

in $[0,T]\times\mathbb{R}^d$, with $V(T,\cdot) = g(\cdot,\mu_T)$ as terminal condition.

Optimal feedback

$$[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \hat{\alpha}\big(t, x, \nu_t, \partial_x V(t,x)\big),$$

Optimal control takes the Markovian form:

$$\hat{\alpha}_t^{\boldsymbol{\nu}} = \tilde{\alpha}\big(t, \hat{X}_t^{\boldsymbol{\nu}}, \nu_t\big), \quad t \in [0,T],$$

for the function $\tilde{\alpha}$

$$\tilde{\alpha}(t,x,\nu) = \hat{\alpha}(t,x,\nu,\partial_x V(t,x))$$

So

$$\mathcal{L}\big(\hat{X}_t^{\boldsymbol{\nu}}, \hat{\alpha}_t^{\boldsymbol{\nu}}\big) = \mathcal{L}\big(\hat{X}_t^{\boldsymbol{\nu}}\big) \circ \big(I_d, \hat{\alpha}(t,\cdot,\nu_t,\partial_x V(t,\cdot))\big)^{-1}.$$

and the equilibrium condition reads:

$$\nu_t = \mathcal{L}\big(\hat{X}_t^{\boldsymbol{\nu}}, \hat{\alpha}_t^{\boldsymbol{\nu}}\big) = \mathcal{L}\big(\hat{X}_t^{\boldsymbol{\nu}}\big) \circ \big(I_d, \hat{\alpha}(t,\cdot,\nu_t,\partial_x V(t,\cdot))\big)^{-1}, \quad t \in [0,T].$$

Finally, the fixed point condition for the flow $\boldsymbol{\nu} = (\nu_t)_{0 \le t \le T}$ can be rewritten as:

$$\begin{cases} \mu_t = \mathcal{L}\big(\hat{X}_t^{\boldsymbol{\nu}}\big), \\ \nu_t = \mu_t \circ \big(I_d, \hat{\alpha}(t,\cdot,\nu_t,\partial_x V(t,\cdot))\big)^{-1}, \end{cases} \qquad t \in [0,T], \tag{5}$$

where μ_t is the first marginal of ν_t on $\mathbb{R}^d$.

Forward-backward PDE system

$$\begin{cases} \partial_t V(t,x) + \dfrac{1}{2}\mathrm{trace}\Big[(\sigma\sigma^\dagger)(t,x,\nu_t)\partial^2_{xx}V(t,x)\Big] \\ \qquad\qquad + H\Big(t,x,\nu_t,\partial_x V(t,x), \hat{\alpha}(t,x,\nu_t,\partial_x V(t,x))\Big) = 0, \\ \partial_t \mu_t - \dfrac{1}{2}\mathrm{trace}\Big[\partial^2_{xx}\Big((\sigma\sigma^\dagger)(t,x,\nu_t)\mu_t\Big)\Big] \\ \qquad\qquad + \mathrm{div}_x\Big(b(t,x,\nu_t,\hat{\alpha}(t,x,\nu_t,\partial_x V(t,x)))\mu_t\Big) = 0, \end{cases}$$

in $[0,T] \times \mathbb{R}^d$, with $V(T,\cdot) = g(\cdot,\mu_T)$ as terminal condition for the first equation, and $\mu_0 = \mathcal{L}(\xi)$ as initial condition for the second equation.

Compared to the classical case, the only **novelty** is the second part of the fixed point

$$\begin{cases} \mu_t = \mathcal{L}(\hat{X}^\nu_t), \\ \nu_t = \mu_t \circ \big(I_d, \hat{\alpha}(t,\cdot,\nu_t,\partial_x V(t,\cdot))\big)^{-1}, \end{cases} \qquad t \in [0,T],$$

where μ_t is the first marginal of ν_t on $\mathbb{R}^d$.

When are these Implicit Expression Uniquely Solvable?

Assume If,

$$\int_{\mathbb{R}^d}\Big(\big[f\big(t,x,\mu\circ(I_d,\psi(\cdot))^{-1},\psi(x)\big)-f\big(t,x,\mu\circ(I_d,\psi'(\cdot))^{-1},\psi(x)\big)\big]$$
$$-\big[f\big(t,x,\mu\circ(I_d,\psi(\cdot))^{-1},\psi'(x)\big)-f\big(t,x,\mu\circ(I_d,\psi'(\cdot))^{-1},\psi'(x)\big)\big]\Big)d\mu(x)$$
$$\geq 0,$$

for $t\in[0,T]$, $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, and ψ and ψ' Borel-measurable from $\mathbb{R}^d$ into A.

Then, t and μ being fixed, **for any** Borel-measurable function

$$\phi:\mathbb{R}^d\to\mathbb{R}^d \qquad \text{in} \qquad L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)$$

there exists a unique square integrable function

$$\psi:\mathbb{R}^d\to A, \quad \text{such that} \quad \nu=\mu\circ(I_d,\psi)^{-1}$$

satisfies

$$\nu=\mu\circ\Big(I_d,\hat{\alpha}\big(t,\cdot,\nu,\phi(\cdot)\big)\Big)^{-1}$$

Examples

- f is of the form
$$f(t,x,\nu,\alpha) = f_0(t,x,\nu) + f_1(t,x,\mu,\alpha),$$
where μ denotes the first marginal of ν on $\mathbb{R}^d$, and smooth $f_0 : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d \times A) \to \mathbb{R}$ and $f_1 : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}$

- f is of the form:
$$f(t,x,\nu,\alpha) = h\big(t,x,\mu,\alpha,Q(x,\cdot)\big),$$
where $h : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \times \mathcal{P}_2(A) \to \mathbb{R}$ isatisfies
$$|h(t,x,\mu,\alpha,\theta)| \le C\big(1+|x|+|\alpha|+M_2(\mu)+M_2(\theta)\big)^2,$$
together with the **Lasry-Lions monotonicity condition**:
$$\forall \theta,\theta' \in \mathcal{P}_2(A), \quad \int_A \big[h(t,x,\mu,\alpha,\theta) - h(t,x,\mu,\alpha,\theta')\big]\, d\big(\theta-\theta'\big)(\alpha) \ge 0.$$
where μ is the first marginal of ν on $\mathbb{R}^d$, and $Q(x,d\alpha)$ is the regular conditional distribution of ν given that the first component

- $f : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d \times A) \times A \to \mathbb{R}$ satisfying:
$$|f(t,x,\nu,\alpha)| \le C\big(1+|x|+|\alpha|+M_2(\nu)\big)^2,$$
together with the **Lasry-Lions monotonicity condition** on the whole $\mathbb{R}^d \times A$, namely:
$$\forall \nu,\nu' \in \mathcal{P}_2(\mathbb{R}^d \times A), \quad \int_{\mathbb{R}^d \times A} \big[f(t,x,\nu,\alpha) - f(t,x,\nu',\alpha)\big]\, d\big(\nu-\nu'\big)(x,\alpha) \ge 0.$$

Technical Results

Probabilistic Approach

Using either

- **BSDE Representation of the Value Function**

or

- **Stochastic Pontryagin Maximum Principle**

together with

- our **monotonicity condition**

EXISTENCE

R.C. - Delarue Chap. 4, Vol. I of big book

If **Lasry-Lions monotonicity condition**

UNIQUENESS

TECHNICAL RESULTS

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BACK TO THE PREVIOUS PRICE IMPACT MODEL

In this particular case:

Reduced Hamiltonian

$$H(t,x,\nu,\alpha,y) = \alpha y + c(\alpha) + c_X(x) - x < h, \theta >$$

so that if we assume $\alpha \mapsto c(\alpha)$ is convex:

$$\hat{\alpha}(t,x,\nu,y) = [c']^{-1}(y)$$

and

$$\tilde{\alpha}(t,x,\nu) = [c']^{-1}(\partial_x V(t,x))$$

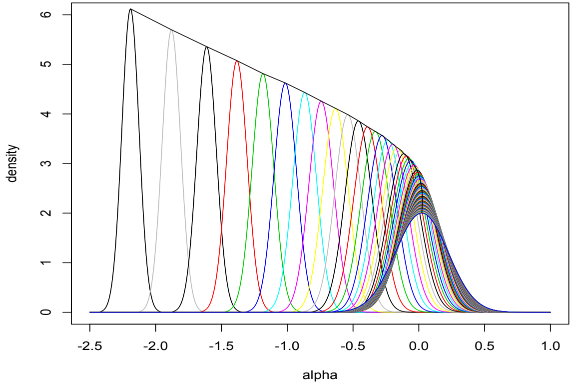
Particular Case:

$$c(\alpha) = \frac{c_\alpha}{2}\alpha^2, \qquad c_X(\alpha) = \frac{c_x}{2}x^2, \qquad h(\alpha) = \bar{h}\alpha, \qquad g(x) = \frac{c_g}{2}x^2$$

Linear Quadratic MFG, explicit solution !

Extended by **Jaimungal-Nourian** and **Cardaliaguet-Lehalle**

Rate of Trading in Equilibrium



Time evolution (from t ranging from 0.06 to $T = 1$) of the marginal density of the optimal rate of trading $\hat{\alpha}_t$ for a representative trader.

Terminal Inventory of a Typical Trader

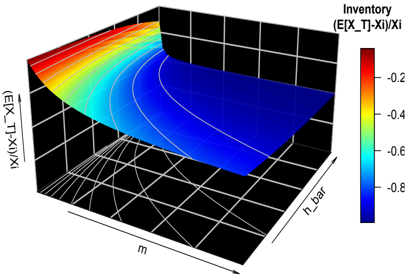


FIGURE: Expected terminal inventory as a function of m and c_X (left) , and as a function of m and $\overline{h}$ (right).

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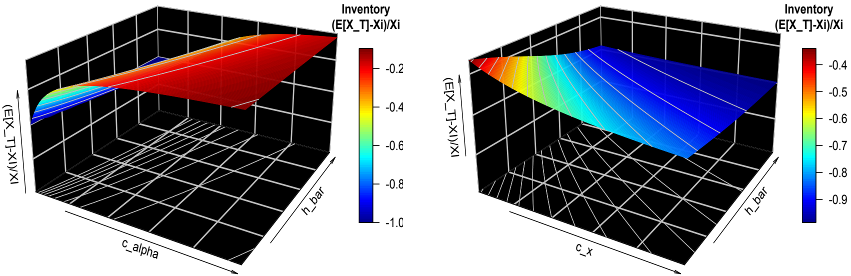

FIGURE: Expected terminal inventory as a function of c_α and $\overline{h}$ (left), and as a function of c_X and $\overline{h}$ (right).

Control of McKean - Vlasov Dynamics

Confusingly **similar** problem !

Solve the **non-standard** *stochastic control problem*

$$\hat{\boldsymbol{\alpha}} = \arg \inf_{\boldsymbol{\alpha} \in \mathbb{A}} \mathbb{E} \left\{ \int_0^T f(t, X_t, \nu_t, \alpha_t) dt + g(X_T, \mu_T) \right\}$$

subject to

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where the measure flow $\boldsymbol{\nu} = (\nu_t)_t$ in $\mathcal{P}(\mathbb{R}^d \times A)$ is actually given by:

$$\forall t \in [0, T], \quad \nu_t = \mathcal{L}(X_t, \alpha_t) \qquad \text{and} \qquad \mu_t = \mathcal{L}(X_t).$$

First studied and solved by **R.C - Delarue** when:

$$\forall t \in [0, T], \quad \nu_t = \mu_t = \mathcal{L}(X_t)$$

Currently, analyzed by **R.C. - Acciaio** when:

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Equivalent to a **causal optimal transport** in path-space from the Wiener measure to the law of a diffusion, but ...... **NB: Pham-Wei**

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Congestion in Crowd Motion

*Trying to understand the effects of **crowded trades**?*

Look at this model of congestion !

Lasry-Lions-Achdou- ....

- bounded domain D in $\mathbb{R}^d$
- exit only possible through $\Gamma \subset \partial D$

$$dX_t^i = \alpha_t^i dt + dW_t^i + dK_t^i, \quad t \in [0, T], \ X_0^i = x_0^i \in D$$

- **reflecting** boundary conditions on $\partial D \setminus \Gamma$
- **Dirichlet** boundary condition on Γ

$$J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\Big[\int_0^{T \wedge \tau^i} \Big(\frac{1}{2}\ell(X_t^i, \mu_t^N)|\alpha_t|^2 + f(t)\Big) dt\Big]$$

- f **penalizes** the time spent in D before the exit
- $\ell(x, \mu)$ models **congestion** around x if μ is the distribution of the individuals (e.g. $\ell(x, \mu) = m(x)^\alpha$)

Congestion & Exit of a Room

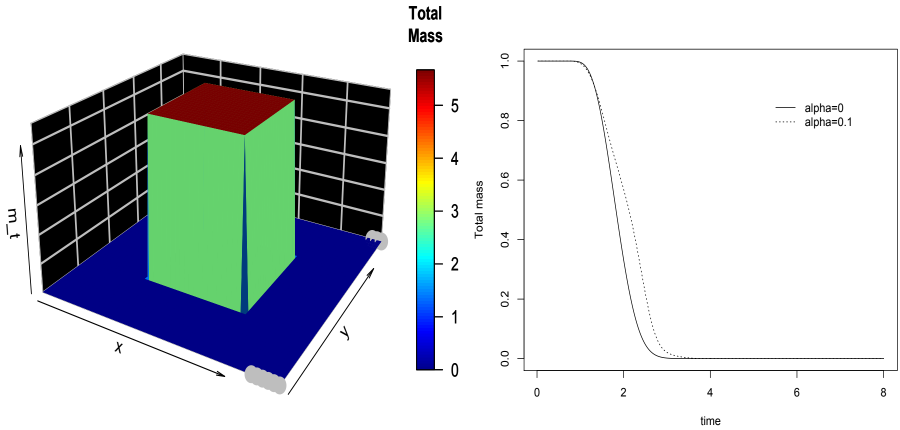
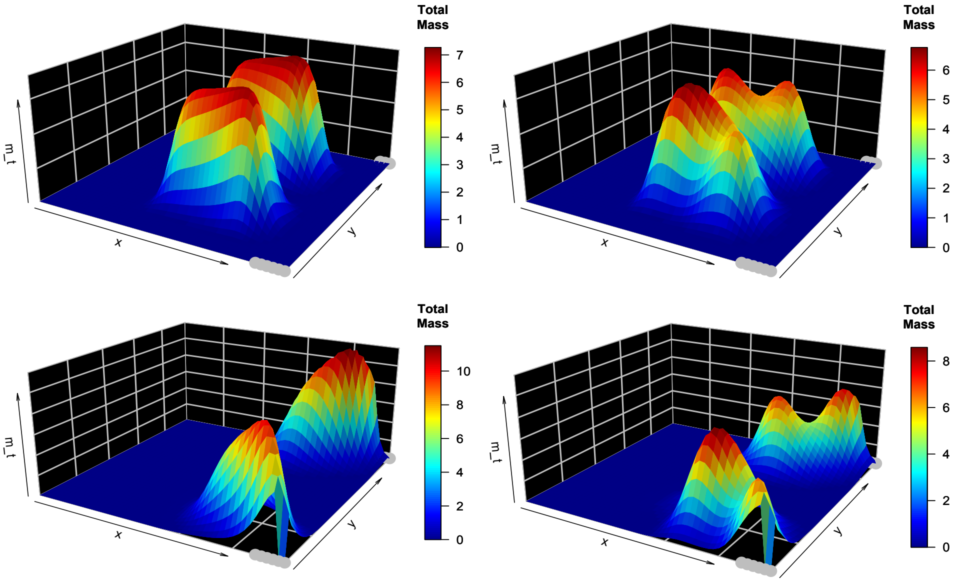
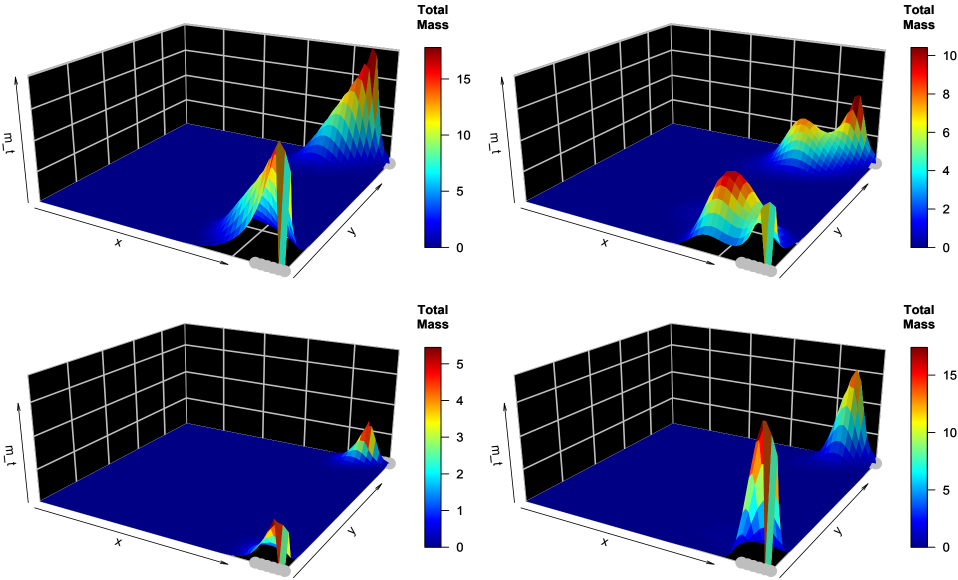


Figure: Left: Initial distribution m_0. Right: Time evolution of the total mass of the distribution m_t of the individuals still in the room at time t without congestion (continuous line) and with moderate congestion (dotted line).

Room Exit Densities



Room Exit Densities



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Putting together

- **Extended Mean Field Game Models**
  (natural for modeling large groups of traders)
- **Congestion MFG Models**
  (with local interactions involving the value of the density at points)

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